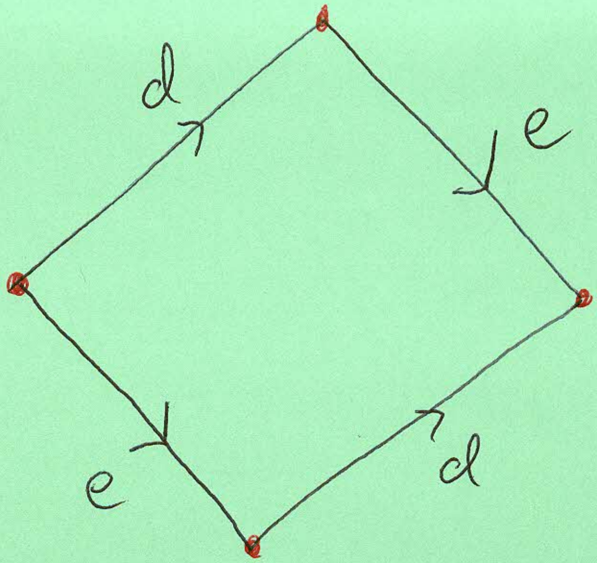
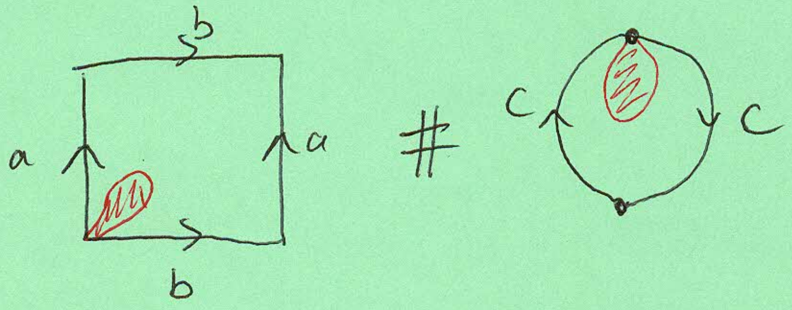


This we recognize as a torus!

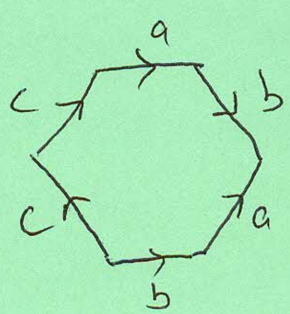
Notice the following: If we take connected sums, we can arrange it so that the word for the connect sum is just the concatenation of the two:

$\mathbb{T} \# \mathbb{P}$



$aba^{-1}b^{-1}$

cc



$aba^{-1}b^{-1}cc$

⚠ Mistake

I think I said on Friday that the test for orientability using the word from the polygon required all vertices to go to the same place. That's not actually necessary. As long a letter does not have an inverse pair, then it is non-orientable.

There are standard forms for the word of any surface:

1) S^2: aa^{-1}

2) $g\mathbb{T}$: $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

3) $m\mathbb{P}$: $a_1 a_1 a_2 a_2 \cdots a_m a_m$

This gives us a restatement of the classification theorem.

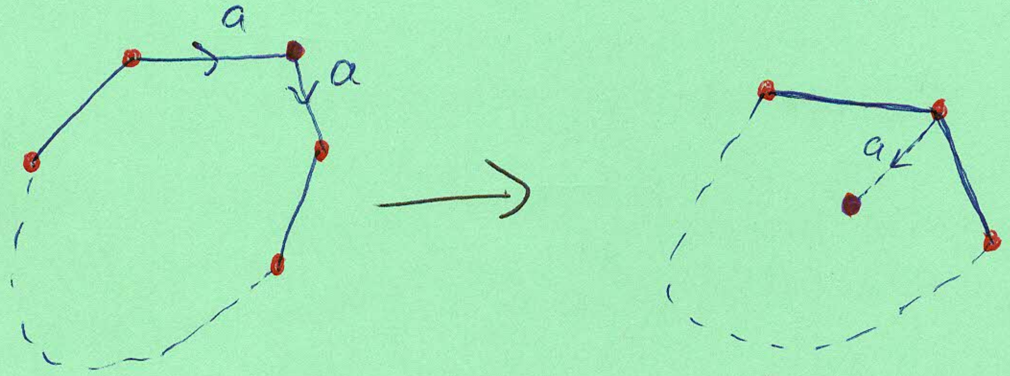
<u>Thm</u>: A plane model for any surface is given by:

- a 2-gon with word aa^{-1},
- a $4g$-gon with word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, or
- a $2m$-gon with word $a_1 a_1 \cdots a_m a_m$

The proof has 6 steps

Step 1: Write the surface as a polygon with edge identifications.

Step 2: Eliminate all adjacent aa^{-1} pairs by gluing:

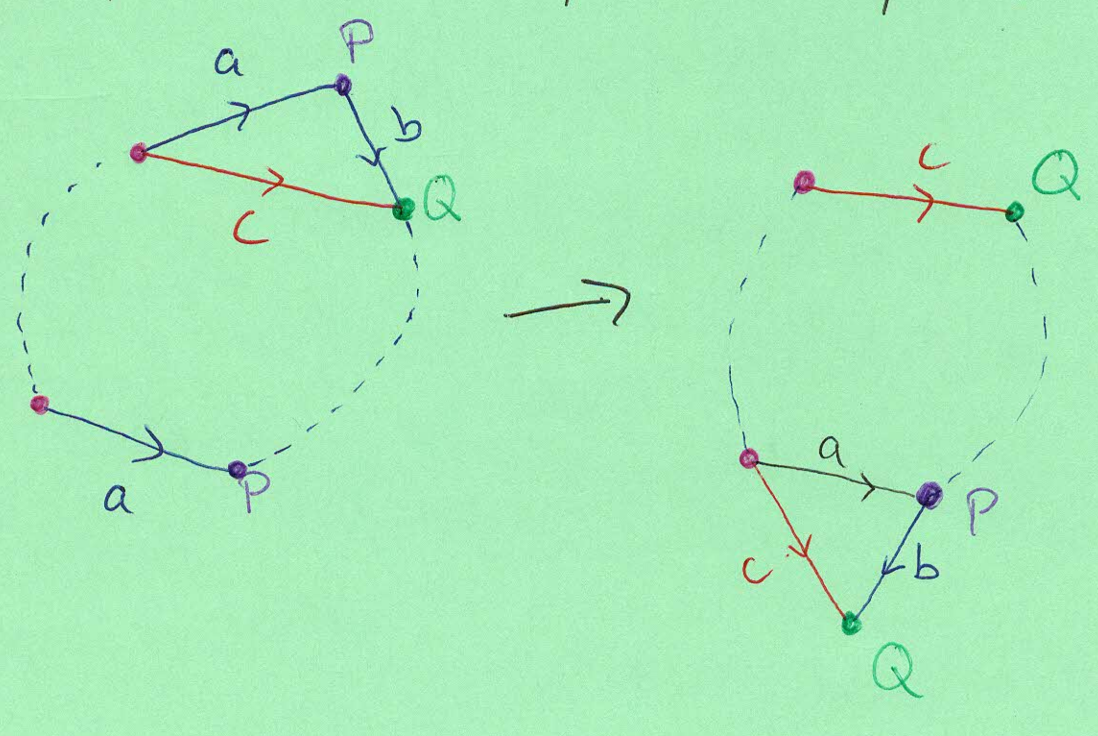


Repeat this step after every subsequent step.

Step 3: Cut and paste to make all vertices go to the same point, or reach a 2-gon with word aa^{-1} (which is a sphere).

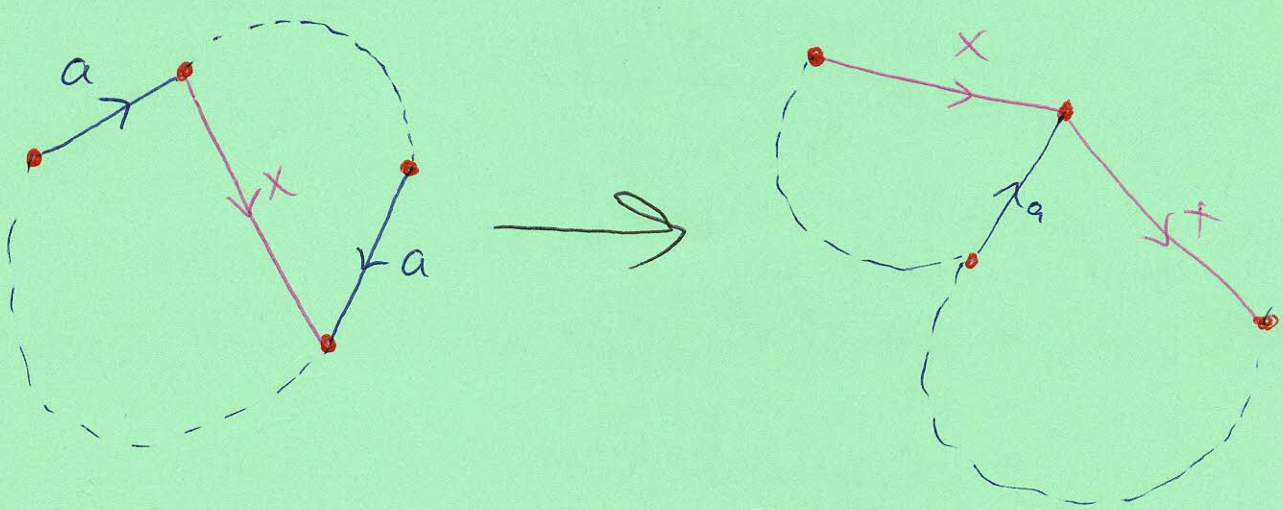
Suppose not all vertices are identified to the same point. Then there are two adjacent which are identified to different points, which we call P & Q. Suppose P is at the tip of an edge a and the tail of an edge b, and that Q is the tip of b.

Draw a new edge, c, from the tail of a to the tip of b, and cut along c. Since there must be another edge a, glue the piece we cut off to a. Since P was at the tip of our first edge a, it also must be at the tip of the second a. Doing this eliminates a P, and creates a Q. Repeating this (and step 2) as necessary, we can make all vertices go to the same place (or end up with a sphere).



Step 4 Cut and paste to make all a⋯a⋯ pairs adjacent.

To do this, draw a new edge, x, from the tip of one a to the tip of the other a. Cut along this edge, then glue along the a's.
Repeat this (and step 2) as needed.



Step 5: After step 4 (and step 2), any pair ⋯a⋯a^{-1}⋯ has a pair ⋯b⋯b^{-1}⋯ alternating with it.
The goal is to make these adjacent.

$\boxed{i}$ Cut from the tail of a to the tail of b, and call this edge x. Gluing along a gives:

$$\cdots x b \cdots x^{-1} \cdots b^{-1}$$

$\boxed{ii}$ Cut from the tail of b to the tail of x^{-1} (i.e., the tip of the second x). Call this new cut y. Glue along b to get

$$\cdots xy\,x^{-1}\cdots y^{-1}$$

$\boxed{iii}$ Cut from the tail of x^{-1} to the tail of y^{-1} and call this cut z^{-1}. Glue along x to get

$$\cdots zyz^{-1}y^{-1}.$$

<u>Step 6</u>: After all this, we end up with a word with adjacent aa pairs and adjacent $aba^{-1}b^{-1}$ pairs. This means the surface is recognizable as some string of connected sums of $\mathbb{P}$'s and $\mathbb{T}$'s, meaning, ultimately, that the surface is

$$k\mathbb{P} \# n\mathbb{T} \text{ for some } k, n \geq 0$$

If $k=0$ but $n>0$, then the surface is $n\mathbb{T}$,
" $n=0$ " $k>0$, " " " " $k\mathbb{P}$,
if $n=k=0$, then the surface is S^2.
If $k, n>0$, then using $\mathbb{P}\#\mathbb{T} = \mathbb{T}\#\mathbb{P} = 3\mathbb{P}$